Multivariable Calculus Formulæ

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Part I
Line Integrals

1 Scalar

When f is the function being integrated, s is arc length, C is the curve to integrate along which is continuous from a to b and $\mathbf{r}$ is the parametrization of C:

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\,dt$$

2 Vector Fields

There are a number of notations for line integrals of vector fields shown below where $\mathbf{F} = \langle f, g, h\rangle$ is the vector field and $\mathbf{T}$ is the unit tangent vector to the curve C which is continuous from a to b and parametrized by $\mathbf{r}$:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt \\ &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_a^b (f\,x'(t) + g\,y'(t) + h\,z'(t))\,dt \\ &= \int_C f\,dx + g\,dy + h\,dz\end{aligned}$$

3 Applications

3.1 Circulation

Circulation is simply expressed as a line integral of a vector field. The circulation of $\mathbf{F}$ on curve C is simply $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

3.2 Flux

Flux is expressed as follows, where $\mathbf{n}$ is the unit normal vector:

$$\int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f\, y'(t) - g\, x'(t))\, dt$$
$$= \int\limits_C f\, dy - g\, dx$$

4 Green's Theorem

4.1 Circulation Form

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{f} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R.

$$\oint\limits_C \mathbf{F} \cdot\, d\mathbf{r} = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

4.1.1 Two-Dimensional Curl

In the double integral from the circulation form formula, the factor of the integral, $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$, is called the two-dimensional curl. This represents the amount of circulation per unit area. Curl can be generalized to $\nabla \times \mathbf{F}$ where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

4.1.2 Calculating Area

Area can be calculated using Green's Theorem. The area of the region R enclosed by curve C can be calculated as such:

$$\begin{aligned} A &= \iint\limits_R dA \\ &= \oint\limits_C x\,dy \\ &= -\oint\limits_C y\,dx \\ &= \frac{1}{2}\oint (x\,dy - y\,dx) \end{aligned}$$

4.2 Flux Form

Under the same conditions as the circulation form where $\mathbf{n}$ is the unit normal vector:

$$\oint\limits_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

4.2.1 Two-dimensional Divergence

In the double integral from the flux form formula, the factor of the integral, $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$, is called the two-dimensional divergence. This measures flux per unit area. Divergence can be generalized to $\nabla \cdot \mathbf{F}$ where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

Part II
Surface Integrals

5 Scalar

Where f is a continuous function on smooth surface S parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $R = \{(u,v)\ a \le u \le b,\ c \le v \le d\}$, $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v}$,

$\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$, then:

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_R f(x(u,v),y(u,v),z(u,v))|\mathbf{t}_u \times \mathbf{t}_v|\,dA$$

If $f(x,y,z) = 1$ then the resulting integral is the surface area of S.
The following shortcut can also be used when the function is explicitly defined as $z = g(x,y)$:

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\,dA$$

6 Vector Fields

Suppose $\mathbf{F} = \langle f,g,h\rangle$ is a continuous vector field on a region of $\mathbf{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,v)$, then:

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\;dA$$

If S is defined as $z = g(x,y)$, then the following shortcut can be used:

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R (-fz_x - gz_y + h)\,dA$$

6.1 Stokes' Theorem

Let S be a smooth oriented surface in $\mathbf{R}^3$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f,g,h\rangle$ is a vector field whose components have continuous first partial derivatives on S.

$$\oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

6.2 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then:

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

where $\mathbf{n}$ is the unit outward normal vector on S.