Calculus Formulæ

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Part I
Calculus and its Applications

Note: a, b, and c are constants, i, x and t are variables and f, g, u and v are differentiable functions. Leibniz notation is occasionally used wherein $f'(x)$ is the derivative of $f(x)$ and $F(x)$ is the antiderivative of $f(x)$. When used in conjunction with one another, x and y refer to the variables of the coordinate plane.

1 Definitions

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{df}{dx} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Note: n and i are variables, c_i is an x value within Δx and Δx_i is the width of the partition or $\frac{b-a}{n}$.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

2 Derivative Calculus

2.1 Basic Derivatives

$$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$$

$$\frac{d}{dx}[c^u] = c^u \ln c\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \frac{\frac{du}{dx}}{u}$$

$$\frac{d}{dx}[\log_c u] = \frac{\frac{du}{dx}}{u \ln c}$$

2.2 Trigonometric Function Derivatives

$$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$$
$$\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx}$$
$$\frac{d}{dx}[\csc u] = -\csc u \cot u \frac{du}{dx}$$
$$\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx}$$
$$\frac{d}{dx}[\cot u] = -\csc^2 u \frac{du}{dx}$$

2.3 Inverse Trigonometric Function Derivatives

$$\frac{d}{dx}[\arcsin u] = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}$$
$$\frac{d}{dx}[\arccos u] = \frac{-\frac{du}{dx}}{\sqrt{1-u^2}}$$
$$\frac{d}{dx}[\arctan u] = \frac{\frac{du}{dx}}{1+u^2}$$
$$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-\frac{du}{dx}}{|u|\sqrt{u^2-1}}$$
$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}$$
$$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-\frac{du}{dx}}{1+u^2}$$

2.4 Complex Derivatives

Product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Chain law:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

3 The Fundamental Theorem of Calculus

3.1 Part One

Function f is continuous over the interval $[a, b]$, then

$$F(x) = \int_a^x f(t)\,dt$$

is thus differentiable over the interval $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

and thus

$$\frac{dF}{dx} = f(x)$$

and in addition, if

$$F(x) = \int_a^{g(x)} f(t)\,dt$$

then

$$\frac{dF}{dx} = f(g(x))\frac{dg}{dx}$$

3.2 Part Two

Function f is continuous over the interval $[a, b]$ and function F is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = F(x)]_a^b = F(b) - F(a)$$

4 Integral Calculus

4.1 Basic Integrals

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int a\,dx = ax + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int \log_a x\,dx = x\log_a x - \frac{x}{\ln a} + c$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + c$$

4.2 Trigonometric Function Integrals

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \csc x\,dx = \log|\csc x - \cot x| + c$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + c$$

$$\int \cot x\,dx = \log|\sin x| + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int -\csc x\cot x\,dx = \csc x + c$$

$$\int \sec x\tan x\,dx = \sec x + c$$

$$\int -\csc^2 x\,dx = \cot x + c$$

4.3 Inverse Trigonometric Function Integrals

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + c$$

$$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2} + c$$

$$\int \arctan x\,dx = x\arctan x - \frac{\ln|1+x^2|}{2} + c$$

$$\int \operatorname{arccsc} x\,dx = x\operatorname{arccsc} x + \operatorname{arcosh} x + c$$

$$\int \operatorname{arcsec} x\,dx = x\operatorname{arcsec} x - \operatorname{arcosh} x + c$$

$$\int \operatorname{arccot} x\,dx = x\operatorname{arccot} x + \frac{\ln|1+x^2|}{2} + c$$

4.4 Complex Integrals

Reverse chain law:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c$$

Reverse derivative of $\ln u$:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Reverse derivative of e^u:

$$\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Definite integration by parts:

$$\int_a^b u\,dv = uv]_a^b - \int_a^b v\,du$$

4.5 Improper Integrals

If $f(x)$ is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If $f(x)$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

If $f(x)$ in continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx =$$

$$\lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

5 Applied Calculus

5.1 Rolle's Theorem

A real valued function f is continuous over $[a, b]$, differentiable over (a, b), and $f(a) = f(b)$, then there exists a value c within (a, b) such that: $f'(c) = 0$.

5.2 Mean Value Theorem

A real valued function f is continuous over $[a, b]$, differentiable over (a, b) and $a < b$, then there exists a value c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.3 Expression of Sums as Integrals

The normal Δx of the integral formula is now $\frac{b-a}{n}$, i is the number partition and $f(a + i\left(\frac{b-a}{n}\right))$ is the height of the partition.

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} f(a+i\left(\frac{b-a}{n}\right)) = \int_a^b f(x)\,dx$$

5.4 Riemann Sum Estimates

$$S = \sum_{i=1}^n (f(y_i)(\Delta x))$$

5.5 L'Hôpital's Rule

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

5.6 Average Value of a Function

Note: a is the beginning and b is the end of the domain and $f(x)$ is the function which you are finding the average value of. M is the average value.

$$M = \frac{1}{b-a} \int_a^b f(x)\, dx$$

5.7 Length of a Curve

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

5.8 Local Linearization

Note: L is the linearization of the function at point a.

$$L = f(a) + f'(a)(x-a)$$

5.9 Taylor Series

Note: b is created based on the desired amount of terms for the series to be drawn out to.

$$\sum_{i=0}^{b} \frac{\frac{d^i f}{dx^i}}{i!}(x-a)^i$$

5.9.1 Specific Maclaurin Series

$$\frac{1}{1-x} = 1 + (x) + (x)^2 + (x)^3 + ... + (x)^n + ...$$

(for $-1 < x < 1$)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ...$$

(for all x)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + \frac{(-1)^n x^{2n}}{(2n)!} + ...$$

(for all x)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

(for all x)

5.10 Euler's Method

Note: a is the interval of the method; the smaller it is, the less error there is.

$$\frac{dy}{dx} = f(t, y(t))$$

$$y(t_0) = y_0$$

$$y_{n+a} = y_n + af(x_n, y_n)$$

5.11 Growth Models

Note: Q is the function for time t, Q_0 is the initial value of Q, k is a proportionality constant, M is the limiting value of Q and A is a constant.

5.11.1 Uninhibited Growth

$$\frac{dQ}{dt} = kQ$$

$$Q = Q_0 e^{kt}$$

$$k > 0$$

5.11.2 Uninhibited Decay

$$\frac{dQ}{dt} = kQ$$

$$Q = Q_0 e^{kt}$$

$$k < 0$$

5.11.3 Inhibited/Logistic Growth

$$\frac{dQ}{dt} = \frac{k}{M} Q(M - Q)$$

$$Q = \frac{M}{1 + Ae^{-kt}}$$

5.11.4 Limited Growth

$$\frac{dQ}{dt} = k(M - Q)$$

$$Q = M - Ae^{-kt}$$

5.12 Descriptions of Change

5.12.1 Actual Absolute Change

$$\Delta y = f(x + \Delta x) - f(a)$$

5.12.2 Estimated Absolute Change

$$dy = f'(a)\,dx$$

5.12.3 Actual Relative Change

$$\frac{\Delta y}{f(a)}$$

5.12.4 Estimated Relative Change

$$\frac{dy}{f(a)}$$

5.12.5 Actual Percent Change

$$\frac{100\Delta y}{f(a)}$$

5.12.6 Estimated Percent Change

$$\frac{100dy}{f(a)}$$

6 Convergence and Divergence

6.1 Convergence and Divergence of Sequences

6.1.1 Determining Convergence or Divergence

To determine whether an integral converges toward a constant or diverges toward infinity, one must try the three steps for testing convergence or divergence:

- Integrate (see Section 4)
- Direct Comparison Test (see 6.1.2 and 6.1.4)
- Limit Comparison Test (see 6.1.3 and 6.1.4)

Note: If a series diverges absolutely, then the series and its absolute value converge. If it converges conditionally, then the series converges but not its absolute value. If it diverges, neither its absolute value or original series converge.

6.1.2 Direct Comparison Test

Let f and g be continuous over the interval $[a, \infty)$ with $0 \leq f(x) \leq g(x)$.

$$\text{If } \int_a^\infty g(x)\, dx \text{ converges,}$$

$$\text{then } \int_a^\infty f(x)\, dx \text{ converges.}$$

$$\text{If } \int_a^\infty f(x)\, dx \text{ diverges,}$$

$$\text{then } \int_a^\infty g(x)\, dx \text{ diverges.}$$

6.1.3 Limit Comparison Test

If the positive functions f and g are continuous over the interval $[a, \infty)$ and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, 0 < L < \infty \text{ then,}$$

$$\int_a^\infty f(x)\, dx \text{ and } \int_a^\infty g(x)\, dx$$

either both converge or both diverge.

6.1.4 P-Series

P-series are three types of series which follow the formula

$$\int_1^b \frac{1}{x^p}\, dx$$

where either $0 < p < 1$, $p = 1$ or $p > 1$. Recognizing functions that fit into the P-series allows one to tell whether or not the integral converges or diverges. Creating P-series based integrals allows one to create a function for comparison in the Direct Comparison or Limit Comparison tests.

$$\text{If } 0 < p < 1, \int_1^b \frac{1}{x^p}\, dx \text{ diverges.}$$

$$\text{If } p = 1, \int_1^b \frac{1}{x}\, dx \text{ diverges.}$$

$$\text{If } p > 1, \int_1^b \frac{1}{x^p}\, dx \text{ converges.}$$

6.2 Convergence and Divergence Tests for Series

6.2.1 Geometric Series Test

Given function:

$$\sum_{k=0}^{\infty} ar^k$$

The series converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

6.2.2 kth-Term Test

If $\lim\limits_{k \to \infty} \neq 0$, the series diverges.

6.2.3 Integral Test

Given function:

$$\sum_{k=1}^{\infty} a_k \text{ where } f(k) = a_k$$

and f is continuous and decreasing and $f(x) \geq 0$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

6.2.4 *p*-series

Given function:

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

The series converges for $p > 1$ and diverges for $P \leq 1$.

6.2.5 Comparison Test

Given functions:

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k, \text{ where } 0 \leq a_k \leq b_k$$

If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

6.2.6 Limit Comparison Test

Given functions:

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=1}^{\infty} b_k$$

where $a_k,\, b_k > 0$, $\lim_{k\to\infty} \frac{a_k}{b_k} = L$. If $0 < L < \infty$, then the two series either both diverge or both converge.

6.2.7 Alternating Series Test

Given function:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ where } a_k > 0 \text{ for all } k.$$

If $\lim\limits_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.

6.2.8 Absolute Convergence Test

For any series with some positive and some negative terms, including alternating series.

If $\sum\limits_{k=1}^{\infty} |a_k|$ converges, then $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely.

6.2.9 Ratio Test

For any series, especially those involving exponents and/or factorials.

For $\lim\limits_{k\to\infty} |\frac{a_{k+1}}{a_k}| = L$, if $L < 1$, $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum\limits_{k=1}^{\infty} a_k$ diverges and if $L = 1$, the test is inconclusive.

6.2.10 Root Test

For any series, especially involving exponents.
For $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$,
if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, the test is inconclusive.

7 Polar Calculus

7.1 Explanation

Polar equations are represented by (r, θ). The r represents a radius drawn out from the origin at angle θ.

7.2 Formulæ

7.2.1 Conversions From Polar to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

7.2.2 Conversions From Rectangular to Polar

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

7.2.3 Slope of a Polar Curve

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\text{If } r = 0 \text{ then } \frac{dy}{dx} = \tan\theta$$

7.2.4 Length of a Polar Curve

$$\int_{\theta_a}^{\theta_b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

7.2.5 Area Under Polar Curve

$$\frac{1}{2}\int_{\theta_a}^{\theta_b} r^2\, d\theta$$

Note: The area between two polar curves is

$$\frac{1}{2}\int_{\theta_a}^{\theta_b} (r_1^2 - r_2^2)\, d\theta$$

and *not*

$$\frac{1}{2}\int_{\theta_a}^{\theta_b} (r_1 - r_2)^2\, d\theta$$

as this would produce an unwanted polynomial.

8 Parametric Calculus

8.1 Explanation

Parametric curves are curves that are defined by two functions, x(t) and y(t).

8.2 Formulæ

8.2.1 Slope of a Parametric Curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ where } \frac{dx}{dt} \neq 0$$

8.2.2 Area Under Parametric Curve

$$\int_{t_a}^{t_b} \mathrm{x}(t)\mathrm{y}'(t)\,dt \text{ or } \int_{t_a}^{t_b} \mathrm{y}(t)\mathrm{x}'(t)\,dt$$

8.2.3 Length of a Parametric Curve

$$\int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

9 Vectors

9.1 Definition

A vector is a magnitude and a vector.

$$A = (x, y)$$

$$B = (x + v_1,\ y + v_2)$$

$$\overrightarrow{AB} = \mathbf{v} = \langle v_1, v_2 \rangle \text{ when } A = (0, 0)$$

$$\text{Magnitude} = |\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$$

9.2 Vector Operations

$$\mathbf{u} \pm \mathbf{v} = \langle u_1 \pm v_1, u_2 \pm v_2 \rangle$$

$$k\mathbf{v} = \langle kv_1, kv_2 \rangle$$

10 Partial Fractions

10.1 Formula

$$\frac{c}{xy} = \frac{a}{x} + \frac{b}{y} \text{ where } ay + bx = c$$

11 Useful Limits

$$\lim_{x\to\infty} \frac{\sin x}{x} = 1$$

$$\lim_{x\to\infty} \frac{1 - \cos x}{x} = 0$$

Part II

Trigonometry Review

12 Definitions and Values

12.1 Trigonometric Function Definitions

$$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{tangent} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\text{cosecant} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\text{secant} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\text{cotangent} = \frac{\text{adjacent}}{\text{opposite}}$$

12.2 Common Trigonometry Values

Radians:	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees:	0°	30°	45°	60°	90°
Sine:	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine:	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent:	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	U
Cosecant:	U	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant:	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	U
Cotangent:	U	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

13 Trigonometric Identities

13.1 The Cofunctions

Note: these definitions are in radians. Alternatively, $\frac{\pi}{2}$ can be represented as 90°.

$$\sin\left[\frac{\pi}{2} - x\right] = \cos x$$
$$\cos\left[\frac{\pi}{2} - x\right] = \sin x$$
$$\csc\left[\frac{\pi}{2} - x\right] = \sec x$$
$$\sec\left[\frac{\pi}{2} - x\right] = \csc x$$
$$\tan\left[\frac{\pi}{2} - x\right] = \cot x$$
$$\cot\left[\frac{\pi}{2} - x\right] = \tan x$$

13.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

13.3 Addition and Subtraction Laws

$$\sin[x+y] = \sin x \cos y + \cos x \sin y$$
$$\sin[x-y] = \sin x \cos y - \cos x \sin y$$
$$\cos[x+y] = \cos x \cos y - \sin x \sin y$$
$$\cos[x-y] = \cos x \cos y + \sin x \sin y$$
$$\tan[x+y] = \frac{\tan x + \tan y}{1 + \tan x \tan y}$$
$$\tan[x-y] = \frac{\tan x - \tan y}{1 - \tan x \tan y}$$

13.4 Double Trigonometric Laws

$$\sin[2x] = 2 \sin x \cos x$$
$$\cos[2x] = 1 - 2\sin^2 x$$
$$\cos[2x] = \cos^2 x - \sin^2 x$$
$$\cos[2x] = 2\cos^2 x - 1$$
$$\tan[2x] = \frac{2\tan x}{1 - \tan^2 x}$$

13.5 Half Trigonometric Laws

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$